Trivariate Mixed Lognormal Distribution: A Statistical Model for Analyzing Cloud Data

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1. Introduction

There is a description (Crow and Shimizu eds., 1988) in which some cloud measurements such as heights, horizontal sizes, and durations are lognormally distributed. Actually this empirical distribution is a distribution conditional on cloud. The mixed lognormal distribution, which consists of a positive probability mass at the origin and a lognormal distribution, is a statistical model which includes the condition of no cloud, just like rain distribution (Kedem and Pavlopoulos, 1991). Since an atmosphere contains three cloud layers, low, medium, and high, we assume that a vector of random variables $(X, Y, Z)'$ representing some cloud measurement in the three cloud layers follows the trivariate mixed lognormal distribution whose definition will be seen in section 3, which is a trivariate extension of the bivariate mixed lognormal distribution defined by Shimizu (1993).

2. Sample

If clouds exist in lower layers, data in upper layers cannot easily be obtained by ground-based observations. Lidar observations from satellites make possible to obtain data in the upper layers. Taking into account this point, suppose that cloud measuremants are observed by radar from ground and space at the same time; we assume that available data have the following structure:

n_1	n_2	n_3	n_4	n_5	n_6	n_7
$x_1, \ldots, x_{n_1}$	$x_1^{1*}, \ldots, x_{n_2}^{1*}$	$0, \ldots, 0$	$x_1^{**}, \ldots, x_{n_4}^{**}$	$0, \ldots, 0$	$0, \ldots, 0$	$0, \ldots, 0$
missing	$y_1^{1*}, \ldots, y_{n_2}^{1*}$	$y_1^{2*}, \ldots, y_{n_3}^{2*}$	$0, \ldots, 0$	$y_1^{**}, \ldots, y_{n_5}^{**}$	$0, \ldots, 0$	$0, \ldots, 0$
$z_1, \ldots, z_{n_1}$	$0, \ldots, 0$	$z_1^{2*}, \ldots, z_{n_3}^{2*}$	$0, \ldots, 0$	$0, \ldots, 0$	$z_1^{**}, \ldots, z_{n_6}^{**}$	$0, \ldots, 0$

where x's, y's, and z's are positive values. A special feature of the structure is that values of y are unobservable if values of x and z are positive.

The following notation is used throughout this paper:

$$n = \sum_{i=1}^{7} n_i,$$

$$\overline{\log x} = \frac{1}{n_1} \sum_{i=1}^{n_1} \log x_i, \quad \overline{\log x^{1*}} = \frac{1}{n_2} \sum_{i=1}^{n_2} \log x_i^{1*}, \quad \overline{\log x^{**}} = \frac{1}{n_4} \sum_{i=1}^{n_4} \log x_i^{**},$$

$$\overline{\log y^{1*}} = \frac{1}{n_2}\sum_{i=1}^{n_2}\log y_i^{1*},\quad \overline{\log y^{2*}} = \frac{1}{n_3}\sum_{i=1}^{n_3}\log y_i^{2*},\quad \overline{\log y^{**}} = \frac{1}{n_5}\sum_{i=1}^{n_5}\log y_i^{**},$$
$$\overline{\log z} = \frac{1}{n_1}\sum_{i=1}^{n_1}\log z_i,\quad \overline{\log z^{2*}} = \frac{1}{n_3}\sum_{i=1}^{n_3}\log z_i^{2*},\quad \overline{\log z^{**}} = \frac{1}{n_6}\sum_{i=1}^{n_6}\log z_i^{**},$$

$$S_x = \sum_{i=1}^{n_1}(\log x_i - \overline{\log x})^2,\quad S_x^{1*} = \sum_{i=1}^{n_2}(\log x_i^{1*} - \overline{\log x^{1*}})^2,\quad S_x^{**} = \sum_{i=1}^{n_4}(\log x_i^{**} - \overline{\log x^{**}})^2,$$
$$S_y^{1*} = \sum_{i=1}^{n_2}(\log y_i^{1*} - \overline{\log y^{1*}})^2,\quad S_y^{2*} = \sum_{i=1}^{n_3}(\log y_i^{2*} - \overline{\log y^{2*}})^2,\quad S_y^{**} = \sum_{i=1}^{n_5}(\log y_i^{**} - \overline{\log y^{**}})^2,$$
$$S_z = \sum_{i=1}^{n_1}(\log z_i - \overline{\log z})^2,\quad S_z^{2*} = \sum_{i=1}^{n_3}(\log z_i^{2*} - \overline{\log z^{2*}})^2,\quad S_z^{**} = \sum_{i=1}^{n_6}(\log z_i^{**} - \overline{\log z^{**}})^2,$$
$$S_{xz} = \sum_{i=1}^{n_1}(\log x_i - \overline{\log x})(\log z_i - \overline{\log z}),\quad S_{xy}^{1*} = \sum_{i=1}^{n_2}(\log x_i^{1*} - \overline{\log x^{1*}})(\log y_i^{1*} - \overline{\log y^{1*}}),$$
$$S_{yz}^{2*} = \sum_{i=1}^{n_3}(\log y_i^{2*} - \overline{\log y^{2*}})(\log z_i^{2*} - \overline{\log z^{2*}}).$$

3. Model

The trivariate mixed lognormal distribution is defined by

$$\begin{aligned}
Pr(X>0,Y>0,Z>0) &= p_{XYZ},\\
Pr(X>0,Y>0,Z=0) &= p_{XYo},\\
Pr(X=0,Y>0,Z>0) &= p_{oYZ},\\
Pr(X>0,Y=0,Z>0) &= p_{XoZ},\\
Pr(X>0,Y=Z=0) &= p_{Xoo},\\
Pr(X=0,Y>0,Z=0) &= p_{oYo},\\
Pr(X=Y=0,Z>0) &= p_{ooZ},\\
Pr(X=Y=Z=0) &= p_{ooo}
\end{aligned}$$

and

$$Pr(X\le x, Y\le y, Z\le z \mid X>0,Y>0,Z>0) = \Lambda_3(x,y,z \mid \mu,\Sigma),$$
$$\mu = (\mu_X,\mu_Y,\mu_Z)',\quad \Sigma = \begin{pmatrix} \sigma_X^2 & \sigma_X\sigma_Y\rho_{XY} & \sigma_X\sigma_Z\rho_{XZ} \\ \sigma_X\sigma_Y\rho_{XY} & \sigma_Y^2 & \sigma_Y\sigma_Z\rho_{YZ} \\ \sigma_X\sigma_Z\rho_{XZ} & \sigma_Y\sigma_Z\rho_{YZ} & \sigma_Z^2 \end{pmatrix},$$
$$Pr(X\le x, Y\le y \mid X>0,Y>0,Z=0) = \Lambda_2(x,y \mid \mu_X^{1*},\mu_Y^{1*},(\sigma_X^{1*})^2,(\sigma_Y^{1*})^2,\rho_{XY}^{1*}),$$
$$Pr(Y\le y, Z\le z \mid X=0,Y>0,Z>0) = \Lambda_2(y,z \mid \mu_Y^{2*},\mu_Z^{2*},(\sigma_Y^{2*})^2,(\sigma_Z^{2*})^2,\rho_{YZ}^{2*}),$$
$$Pr(X\le x, Z\le z \mid X>0,Y=0,Z>0) = \Lambda_2(x,z \mid \mu_X^{3*},\mu_Z^{3*},(\sigma_X^{3*})^2,(\sigma_Z^{3*})^2,\rho_{XZ}^{3*}),$$
$$Pr(X\le x \mid X>0,Y=Z=0) = \Lambda(x \mid \mu_X^{**},(\sigma_X^{**})^2),$$
$$Pr(Y\le y \mid X=0,Y>0,Z=0) = \Lambda(y \mid \mu_Y^{**},(\sigma_Y^{**})^2),$$
$$Pr(Z\le z \mid X=Y=0,Z>0) = \Lambda(z \mid \mu_Z^{**},(\sigma_Z^{**})^2)$$

for $x,y,z>0$, where $\Lambda(t \mid \mu_T^{**},(\sigma_T^{**})^2)$ is the distribution function of a univariate lognormal distribution $\Lambda(\mu_T^{**},(\sigma_T^{**})^2)$, $\Lambda_2(s,t \mid \mu_S^{k*},\mu_T^{k*},(\sigma_S^{k*})^2,(\sigma_T^{k*})^2,\rho_{ST}^{k*})$ the joint distribution function of a bivariate lognormal distribution $\Lambda_2(\mu_S^{k*},\mu_T^{k*},(\sigma_S^{k*})^2,(\sigma_T^{k*})^2,\ \rho_{ST}^{k*})$, and

$\Lambda_3(x,y,z \mid \mu,\Sigma)$ the joint distribution function of a trivariate lognormal disribution $\Lambda_3(\mu,\Sigma)$. Then, for example, for $x \geq 0$,

$$\begin{aligned} Pr(X \leq x) &= \delta_X + p_{X00}\Lambda(x \mid \mu_X^{**},(\sigma_X^{**})^2) + p_{X0Z}\Lambda(x \mid \mu_X^{3*},(\sigma_X^{3*})^2) \\ &\quad + p_{XY0}\Lambda(x \mid \mu_X^{1*},(\sigma_X^{1*})^2) + p_{XYZ}\Lambda(x \mid \mu_X,\sigma_X^2), \end{aligned}$$

where $\delta_X = Pr(X=0) = p_{000} + p_{00Z} + p_{0Y0} + p_{0YZ}$, which is a mixed distribution whose continuous part consists of a mixture of four lognormal distributions. If $\mu_X = \mu_X^{1*} = \mu_X^{3*} = \mu_X^{**}$ and $\sigma_X^2 = (\sigma_X^{1*})^2 = (\sigma_X^{3*})^2 = (\sigma_X^{**})^2$, then it clearly holds that

$$Pr(X \leq x) = \delta_X + (1-\delta_X)\Lambda(x \mid \mu_X,\sigma_X^2),$$

which is a mixed lognormal distribution. Bivariate distribution functions are also obtainable. For example, for $x, y \geq 0$,

$$\begin{aligned} &Pr(X \leq x, Y \leq y) = \delta_{XY} + p_{X00}\Lambda(x \mid \mu_X^{**},(\sigma_X^{**})^2) + p_{X0Z}\Lambda(x \mid \mu_X^{3*},(\sigma_X^{3*})^2) \\ &\quad + p_{0Y0}\Lambda(y \mid \mu_Y^{**},(\sigma_Y^{**})^2) + p_{0YZ}\Lambda(y \mid \mu_Y^{2*},(\sigma_Y^{2*})^2) \\ &\quad + p_{XY0}\Lambda_2(x,y \mid \mu_X^{1*},\mu_Y^{1*},(\sigma_X^{1*})^2,(\sigma_Y^{1*})^2,\rho_{XY}^{1*}) + p_{XYZ}\Lambda_2(x,y \mid \mu_X,\mu_Y,\sigma_X^2,\sigma_Y^2,\rho_{XY}), \end{aligned}$$

where $\delta_{XY} = Pr(X=Y=0) = p_{000} + p_{00Z}$.

Some other quantities of interest are

$$\begin{aligned} Pr(Y \leq y \mid X>0, Z>0) &= \frac{1}{p_{X0Z}+p_{XYZ}}\{p_{X0Z} + p_{XYZ}\Lambda(y \mid \mu_Y,\sigma_Y^2)\}, \\ E(Y \mid X>0, Z>0) &= \frac{p_{XYZ}}{p_{X0Z}+p_{XYZ}}\exp[\mu_Y + \frac{1}{2}\sigma_Y^2], \\ Var(Y \mid X>0, Z>0) &= \frac{p_{XYZ}}{p_{X0Z}+p_{XYZ}}\exp[2\mu_Y + \sigma_Y^2]\left(e^{\sigma_Y^2} - \frac{p_{XYZ}}{p_{X0Z}+p_{XYZ}}\right). \end{aligned}$$

4. Maximum likelihood (ML) estimation

The likelihood function is expressed as

$$L = L_1 \times L_2,$$

where

$$\begin{aligned} L_1 &= \binom{n}{n_1,n_2,n_3,n_4,n_5,n_6,n_7}(p_{XYZ}+p_{X0Z})^{n_1}p_{XY0}^{n_2}p_{0YZ}^{n_3}p_{X00}^{n_4}p_{0Y0}^{n_5}p_{00Z}^{n_6}p_{000}^{n_7} \\ L_2 &= \prod_{i=1}^{n_1}\frac{p_{XYZ}\lambda_2(\mu_X,\mu_Z,\sigma_X^2,\sigma_Z^2,\rho_{XZ} \mid x_i,z_i) + p_{X0Z}\lambda_2(\mu_X^{3*},\mu_Z^{3*},(\sigma_X^{3*})^2,(\sigma_Z^{3*})^2,\rho_{XZ}^{3*} \mid x_i,z_i)}{p_{XYZ}+p_{X0Z}} \\ &\times \prod_{i=1}^{n_2}\lambda_2(\mu_X^{1*},\mu_Y^{1*},(\sigma_X^{1*})^2,(\sigma_Y^{1*})^2,\rho_{XY}^{1*} \mid x_i^{1*},y_i^{1*})\prod_{i=1}^{n_3}\lambda_2(\mu_Y^{2*},\mu_Z^{2*},(\sigma_Y^{2*})^2,(\sigma_Z^{2*})^2,\rho_{YZ}^{2*} \mid y_i^{2*},z_i^{2*}) \\ &\times \prod_{i=1}^{n_4}\lambda(\mu_X^{**},(\sigma_X^{**})^2 \mid x_i^{**})\prod_{i=1}^{n_5}\lambda(\mu_Y^{**},(\sigma_Y^{**})^2 \mid y_i^{**})\prod_{i=1}^{n_6}\lambda(\mu_Z^{**},(\sigma_Z^{**})^2 \mid z_i^{**}). \end{aligned}$$

Here $\binom{n}{n_1, n_2, n_3, n_4, n_5, n_6, n_7}$ indicates a multinomial coefficient, $\lambda(\mu_T^{**}, (\sigma_T^{**})^2 \mid t)$ the density function of $\Lambda(\mu_T^{**}, (\sigma_T^{**})^2)$, and $\lambda_2(\mu_S^{**}, \mu_T^{**}, (\sigma_S^{**})^2, (\sigma_T^{**})^2, \rho_{ST}^{**} \mid s, t)$ the joint density function of $\Lambda_2(\mu_S^{**}, \mu_T^{**}, (\sigma_S^{**})^2, (\sigma_T^{**})^2, \rho_{ST}^{**})$.

From now on we assume that $\mu_x = \mu_X^{3*}$, $\mu_z = \mu_Z^{3*}$, $\sigma_x = \sigma_X^{3*}$, $\sigma_z = \sigma_Z^{3*}$, $\rho_{xz} = \rho_{xz}^{3*}$, i.e., the marginal distribution of X and Z when $X > 0$ and $Z > 0$ is the same bivariate lognormal distribution whether Y is positive or not, and write $p = p_{XYZ} + p_{Xoz}$. Then the ML estimates of $p, p_{XYo}, p_{oYZ}, p_{Xoo}, p_{oYo}, p_{ooZ}$ and p_{ooo} are given by

$$\hat{p} = \frac{n_1}{n},\ \hat{p}_{XYo} = \frac{n_2}{n},\ \hat{p}_{oYZ} = \frac{n_3}{n},\ \hat{p}_{Xoo} = \frac{n_4}{n},\ \hat{p}_{oYo} = \frac{n_5}{n},\ \hat{p}_{ooZ} = \frac{n_6}{n},\ \hat{p}_{ooo} = \frac{n_7}{n}.$$

Making an assumption

(1) $p_{XYZ} = \max(0, p_{XYo} + p_{oYZ} + p_{XoZ} - 1)$

or (2) $p_{XYZ} = \min(p_{Xoo}, p_{oYo}, p_{ooZ})$

as a constraint, we can have estimates of p_{XYZ}, p_{XoZ}. This assumption is taken from (1) the minimum overlap assumption or (2) the maximum overlap assumption (Morcrette and Fouquart, 1986; Tian and Curry, 1989). The assumption (2) is meaningful only if $n_1 \geq \min(n_4, n_5, n_6)$.

We use the following notation on hypotheses:

- $H_X^\mu : \mu_x = \mu_X^{1*} = \mu_X^{**} (= \mu_1)$,
- $H_Y^\mu : \mu_y = \mu_Y^{1*} = \mu_Y^{2*} = \mu_Y^{**} (= \mu_2)$,
- $H_Z^\mu : \mu_z = \mu_Z^{2*} = \mu_Z^{**} (= \mu_3)$,
- $H_X^\sigma : \sigma_x = \sigma_X^{1*} = \sigma_X^{**} (= \sigma_1)$,
- $H_Y^\sigma : \sigma_Y^{1*} = \sigma_Y^{2*} = \sigma_Y^{**} (= \sigma_2)$,
- $H_Z^\sigma : \sigma_z = \sigma_Z^{2*} = \sigma_Z^{**} (= \sigma_3)$,
- $H^\sigma : \sigma_x = \sigma_X^{1*} = \sigma_X^{**} = \sigma_Y^{1*} = \sigma_Y^{2*} = \sigma_Y^{**} = \sigma_z = \sigma_Z^{2*} = \sigma_Z^{**} (= \sigma)$,
- $H^\rho : \rho_{xz} = \rho_{XY}^{1*} = \rho_{YZ}^{2*} (= \rho)$.

Shimizu (1993) points out that the main concerned parameter of $\Lambda(\mu, \sigma^2)$ is σ, a shape parameter, while $\exp(\mu)$ is a scale parameter, and that the constancy of σ , which means "structural stability" in a sense, can be seen in daily rainfall distributions as well as in many lognormal applications. Thus, the following cases may be considered to be more important hypotheses:

Case 1 (General; all parameter values are different.), **Case 2** (H_X^σ), **Case 3** (H_Y^σ), **Case 4** (H_Z^σ), **Case 5** (H_X^σ, H_Y^σ), **Case 6** (H_X^σ, H_Z^σ), **Case 7** (H_Y^σ, H_Z^σ), **Case 8** (H^σ), **Case 9** (H_X^σ, H^ρ), **Case 10** (H_Y^σ, H^ρ), **Case 11** (H_Z^σ, H^ρ), **Case 12** ($H_X^\sigma, H_Y^\sigma, H^\rho$), **Case 13** ($H_X^\sigma, H_Z^\sigma, H^\rho$), **Case 14** ($H_Y^\sigma, H_Z^\sigma, H^\rho$), **Case 15** ($H^\sigma, H^\rho$).

Others are, for example, $(H_Y^\mu, H^\sigma, H^\rho)$ and $(H_X^\mu, H_Y^\mu, H_Z^\mu, H^\sigma, H^\rho)$. These hypotheses include constraints on μ's.

The log-likelihood for estimating μ's, σ's and ρ's is

$$\ell = -\Big[n_1\left(\log\sigma_X + \log\sigma_Z + \tfrac{1}{2}\log(1-\rho_{XZ}^2)\right) + n_2\left(\log\sigma_X^{1*} + \log\sigma_Y^{1*} + \tfrac{1}{2}\log(1-(\rho_{XY}^{1*})^2)\right)$$
$$+n_3\left(\log\sigma_Y^{2*} + \log\sigma_Z^{2*} + \tfrac{1}{2}\log(1-(\rho_{YZ}^{2*})^2)\right) + n_4\log\sigma_X^{**} + n_5\log\sigma_Y^{**} + n_6\log\sigma_Z^{**}$$
$$+\frac{1}{2(1-\rho_{XZ}^2)}\sum_{i=1}^{n_1}\left\{\left(\frac{\log x_i-\mu_X}{\sigma_X}\right)^2 - 2\rho_{XZ}\left(\frac{\log x_i-\mu_X}{\sigma_X}\right)\left(\frac{\log z_i-\mu_Z}{\sigma_Z}\right) + \left(\frac{\log z_i-\mu_Z}{\sigma_Z}\right)^2\right\}$$
$$+\frac{1}{2(1-(\rho_{XY}^{1*})^2)}\sum_{i=1}^{n_2}\left\{\left(\frac{\log x_i^{1*}-\mu_X^{1*}}{\sigma_X^{1*}}\right)^2 - 2\rho_{XY}^{1*}\left(\frac{\log x_i^{1*}-\mu_X^{1*}}{\sigma_X^{1*}}\right)\left(\frac{\log y_i^{1*}-\mu_Y^{1*}}{\sigma_Y^{1*}}\right) + \left(\frac{\log y_i^{1*}-\mu_Y^{1*}}{\sigma_Y^{1*}}\right)^2\right\} \quad (1)$$
$$+\frac{1}{2(1-(\rho_{YZ}^{2*})^2)}\sum_{i=1}^{n_3}\left\{\left(\frac{\log y_i^{2*}-\mu_Y^{2*}}{\sigma_Y^{2*}}\right)^2 - 2\rho_{YZ}^{2*}\left(\frac{\log y_i^{2*}-\mu_Y^{2*}}{\sigma_Y^{2*}}\right)\left(\frac{\log z_i^{2*}-\mu_Z^{2*}}{\sigma_Z^{2*}}\right) + \left(\frac{\log z_i^{2*}-\mu_Z^{2*}}{\sigma_Z^{2*}}\right)^2\right\}$$
$$+\frac{1}{2(\sigma_X^{**})^2}\sum_{i=1}^{n_4}(\log x_i^{**}-\mu_X^{**})^2 + \frac{1}{2(\sigma_Y^{**})^2}\sum_{i=1}^{n_5}(\log y_i^{**}-\mu_Y^{**})^2 + \frac{1}{2(\sigma_Z^{**})^2}\sum_{i=1}^{n_6}(\log z_i^{**}-\mu_Z^{**})^2\Big].$$

In Case 1 μ's, σ's and ρ's can be estimated by

$$\left.\begin{array}{l} \hat\mu_X = \overline{\log x},\quad \hat\mu_X^{1*} = \overline{\log x^{1*}},\quad \hat\mu_X^{**} = \overline{\log x^{**}},\\ \hat\mu_Y^{1*} = \overline{\log y^{1*}},\quad \hat\mu_Y^{2*} = \overline{\log y^{2*}},\quad \hat\mu_Y^{**} = \overline{\log y^{**}},\\ \hat\mu_Z = \overline{\log z},\quad \hat\mu_Z^{2*} = \overline{\log z^{2*}},\quad \hat\mu_Z^{**} = \overline{\log z^{**}}, \end{array}\right\} \quad (2)$$

$$\left.\begin{array}{l} \hat\sigma_X^2 = \frac{1}{n_1}S_x,\quad (\hat\sigma_X^{1*})^2 = \frac{1}{n_2}S_x^{1*},\quad (\hat\sigma_X^{**})^2 = \frac{1}{n_4}S_x^{2*},\\ (\hat\sigma_Y^{1*})^2 = \frac{1}{n_2}S_y^{1*},\quad (\hat\sigma_Y^{2*})^2 = \frac{1}{n_3}S_x^{2*},\quad (\hat\sigma_Y^{**})^2 = \frac{1}{n_5}S_x^{**},\\ \hat\sigma_Z^2 = \frac{1}{n_1}S_z,\quad (\hat\sigma_Z^{2*})^2 = \frac{1}{n_3}S_z^{2*},\quad (\hat\sigma_Z^{**})^2 = \frac{1}{n_6}S_z^{**}, \end{array}\right\} \quad (3)$$

$$\hat\rho_{XZ} = \frac{S_{xz}}{\sqrt{S_xS_y}},\quad \hat\rho_{XY}^{1*} = \frac{S_{xy}^{1*}}{\sqrt{S_x^{1*}S_y^{1*}}},\quad \hat\rho_{YZ}^{2*} = \frac{S_{yz}^{2*}}{\sqrt{S_x^{2*}S_y^{2*}}}. \quad (4)$$

Since there is no observation that triple values are all positive, parameters μ_Y, σ_Y^2, ρ_{XY} and ρ_{YZ} are inestimable unless there is any assumption. We can use the following ad hoc estimates in all cases:

$$\hat\mu_Y = \frac{1}{n_2+n_3+n_5}(n_2\hat\mu_Y^{1*} + n_3\hat\mu_Y^{2*} + n_5\hat\mu_Y^{**}),$$
$$\hat\sigma_Y^2 = \frac{1}{n_2+n_3+n_5}(S_y^{1*} + S_y^{2*} + S_y^{**}),$$
$$\hat\rho_{XY} = \hat\rho_{XY}^{1*},\quad \hat\rho_{YZ} = \hat\rho_{YZ}^{2*}.$$

When there are no constraints on μ's, the estimates are given by (2). As for estimates of σ's and ρ's, estimating functions can be obtained in principle. For example, the ML estimates of σ and ρ in **Case 15** (H^σ, H^ρ) satisfy the estimating functions

$$\{2(n_1+n_2+n_3)+n_4+n_5+n_6\}\hat\sigma^2 =$$
$$\frac{1}{1-\hat\rho^2}\left\{S_x + S_z + S_x^{1*} + S_y^{1*} + S_y^{2*} + S_z^{2*} - 2\hat\rho(S_{xz} + S_{xy}^{1*} + S_{yz}^{2*})\right\}$$
$$+S_x^{**} + S_y^{**} + S_z^{**}$$

and

$$
\begin{aligned}
&(n_1+n_2+n_3)(S_x^{**}+S_y^{**}+S_z^{**})\hat{\rho}^3 \\
&\quad -(n_4+n_5+n_6)(S_{xz}+S_{xy}^{1*}+S_{yz}^{2*})\hat{\rho}^2 \\
&\quad +\{(n_1+n_2+n_3+n_4+n_5+n_6)(S_x+S_z+S_x^{1*}+S_y^{1*}+S_y^{2*}+S_z^{2*}) \\
&\quad -(n_1+n_2+n_3)(S_x^{**}+S_y^{**}+S_z^{**})\}\hat{\rho} \\
&\quad -\{2(n_1+n_2+n_3)+n_4+n_5+n_6\}(S_{xz}+S_{xy}^{1*}+S_{yz}^{2*})=0.
\end{aligned}
$$

However, for σ's and ρ's, denoted by $(\theta_1, ..., \theta_k)$, it is enough for getting ML estimates to solve the following simultaneous equations numerically:

$$
\frac{\partial \ell}{\partial \theta_i} = 0 \quad (i = 1, ..., k),
$$

where ℓ is given by (1) with (2).

In the case of $(H_Y^\mu, H^\sigma, H^\rho)$, we have the ML estimates and estimating function

$$
\hat{\mu}_X = \overline{\log x}, \quad \hat{\mu}_X^{**} = \overline{\log x^{**}}, \quad \hat{\mu}_Z = \overline{\log z}, \quad \hat{\mu}_Z^{**} = \overline{\log z^{**}},
$$

$$
\hat{\mu}_X^{1*} = \overline{\log x^{1*}} - \hat{\rho}(\overline{\log y^{1*}} - \hat{\mu}_Y), \quad \hat{\mu}_Z^{2*} = \overline{\log z^{2*}} - \hat{\rho}(\overline{\log y^{2*}} - \hat{\mu}_Y),
$$

$$
\hat{\mu}_Y = \frac{1}{n_2+n_3+n_5}(n_2\overline{\log y^{1*}} + n_3\overline{\log y^{2*}} + n_5\overline{\log y^{**}}),
$$

$$
\begin{aligned}
&\{2(n_1+n_2+n_3)+n_4+n_5+n_6\}\hat{\sigma}^2 = \\
&\quad \frac{1}{1-\hat{\rho}^2}\left\{S_x+S_z+S_x^{1*}+S_y^{1*}+S_y^{2*}+S_z^{2*}-2\hat{\rho}(S_{xz}+S_{xy}^{1*}+S_{yz}^{2*})\right\} \\
&\quad +S_x^{**}+S_y^{**}+S_z^{**}+n_2(\overline{\log y^{1*}}-\hat{\mu}_Y)^2+n_3(\overline{\log y^{2*}}-\hat{\mu}_Y)^2+n_5(\overline{\log y^{**}}-\hat{\mu}_Y)^2,
\end{aligned}
$$

$$
\begin{aligned}
&(n_1+n_2+n_3)\{S_x^{**}+S_y^{**}+S_z^{**} \\
&\quad +n_2(\overline{\log y^{1*}}-\hat{\mu}_Y)^2+n_3(\overline{\log y^{2*}}-\hat{\mu}_Y)^2+n_5(\overline{\log y^{**}}-\hat{\mu}_Y)^2\}\hat{\rho}^3 \\
&\quad -(n_4+n_5+n_6)(S_{xz}+S_{xy}^{1*}+S_{yz}^{2*})\hat{\rho}^2 \\
&\quad +[(n_1+n_2+n_3+n_4+n_5+n_6)(S_x+S_z+S_x^{1*}+S_y^{1*}+S_y^{2*}+S_z^{2*}) \\
&\quad -(n_1+n_2+n_3)\{S_x^{**}+S_y^{**}+S_z^{**} \\
&\quad +n_2(\overline{\log y^{1*}}-\hat{\mu}_Y)^2+n_3(\overline{\log y^{2*}}-\hat{\mu}_Y)^2+n_5(\overline{\log y^{**}}-\hat{\mu}_Y)^2\}]\hat{\rho} \\
&\quad -\{2(n_1+n_2+n_3)+n_4+n_5+n_6\}(S_{xz}+S_{xy}^{1*}+S_{yz}^{2*})=0.
\end{aligned}
$$

However, when there are constraints on μ's, there are cases where it is difficult to have ML estimates even for μ's in closed forms. Thus, it is recommended to obtain ML estimates for μ's, σ's and ρ's, denoted by $(\theta_1, ..., \theta_m)$, by numerically solving simultaneous equations

$$\frac{\partial \ell}{\partial \theta_i} = 0 \quad (i = 1, ..., m).$$

5. Model selection

A procedure which minimizes Akaike's Information Criterion (AIC),

$$\mathrm{AIC} = -2\hat{\ell} + 2m,$$

can be used to select an "optimal" model from among many, where $\hat{\ell}$ is the maximum loglikelihood, which is computed by replacing the parameters by their maximum likelihood estimates in (1), and m the number of free parameters in the model. The procedure is much different from testing statistical hypotheses because it is free from assigning a practical level of significance. Sakamoto et al. (1983) have given remarks on the practical use of the minimum AIC procedure: The number of free parameters should be less than $2\sqrt{N}$ ($N/2$ at most), where $N = \sum_{i=1}^{ns} n_i$ in our case, and the difference of AIC's is considered to be significant if it is larger than 1-2.

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